ON THE SIZE OF NIKODYM SETS IN FINITE FIELDS

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ABSTRACT. Let $\mathbb{F}_q$ denote a finite field of q elements. Define a set $B \subset \mathbb{F}_q^n$ to be Nikodym if for each $x \in B^c$, there exists a line L such that $L \cap B^c = \{x\}$. The main purpose of this note is to show that the size of every Nikodym set is at least $C_n \cdot q^n$, where C_n depends only on n.

1. INTRODUCTION

The finite field Kakeya problem, posed by Wolff in his influential survey [13], asks for the smallest subset of $\mathbb{F}_q^n$ that contains a line in each direction, where $\mathbb{F}_q$ denotes a finite field of q elements. A subset containing a line in each direction is called a Kakeya set. In analogy with the Euclidean Kakeya problem, Wolff conjectured that $\sharp K \geq C_n q^n$ holds for any Kakeya set $K \subset \mathbb{F}_q^n$, where C_n depends only on the dimension n. For $n = 2$ Wolff immediately proved the bound $\sharp K \geq q(q+1)/2$, and it is best possible when q is even. To the author's knowledge, Blokhuis and Mazzocca [1] studied the finite field Kakeya problem in two dimensions and proved the sharp bound $\sharp K \geq q(q+1)/2 + (q-1)/2$ when q is odd, as conjectured by Faber in [6]. The higher dimensional finite field Kakeya problem has been extensively investigated in [2, 8, 10, 12, 13] such as proving the bound $\sharp K \geq C_n q^{(n+2)/2}$ or $\sharp K \geq C_n q^{(4n+3)/7}$. Recently, using the polynomial method in algebraic extremal combinatorics, Dvir [5] completely confirmed this conjecture by proving

$$\sharp K \geq \binom{n+q-1}{n}.$$

On the other hand, Nikodym [9] proved that there exists a null set in the unit square such that every point of the complement is "linearly accessible through the set", which means it lies on a line that is otherwise included in the set. Falconer [7] extended Nikodym's result to higher dimensions proving there exists a set $N \subset \mathbb{R}^n$ of zero Lebesgue measure such that for each $x \in N^c$, there is a hyperplane P satisfying $P \cap N^c = \{x\}$. In the Euclidean spaces Nikodym sets are closely related to Kakeya sets through Carbery's transformation [4, 11].

Motivated by the above works, we shall define a set B in $\mathbb{F}_q^n$ to be Nikodym if for each $x \in B^c$ there exists a line L such that $L \cap B^c = \{x\}$. The main purpose of this note is to prove the lower bound

$$\sharp B \geq \binom{n+q-2}{n}.$$

Slightly different with the two dimensional finite field Kakeya problem, this bound is not best possible in two dimensions.

2. General dimensions

Theorem 2.1. *Any Nikodym set $B \subset \mathbb{F}_q^n$ satisfies*

$$|B| \geq \binom{n+q-2}{n},$$

where $\mathbb{F}_q$ denotes a finite field of q elements.

Proof. We argue by contradiction and suppose

$$|B| < \binom{n+q-2}{n}.$$

A basic result in combinatorics [3] says that the number of monomials in $\mathbb{F}[x_1, \ldots, x_n]$ of degree at most d is

$$\binom{n+d}{n},$$

hence there exists a nonzero polynomial $g \in \mathbb{F}[x_1, \ldots, x_n]$ of degree at most $q-2$ such that

$$g(y) = 0 \quad (\forall y \in B).$$

For each $x \in B^c$, there exists a line L such that

$$L \cap B^c = \{x\}.$$

The restriction of g to this line is a univariate polynomial of degree at most $q-2$, and since it has at least $q-1$ zeros, it must be zero on the entire line L. Considering x belongs to this line, it follows that

$$g(x) = 0.$$

This would mean g is the zero polynomial, a contradiction.

□

3. Two dimensions

Theorem 3.1. *Any Nikodym set $B \subset \mathbb{F}_q^2$ satisfies*

$$\sharp B \geq \frac{2q^2}{3} + O(q) \quad (q \to \infty),$$

where $\mathbb{F}_q$ denotes a finite field of q elements.

Proof. Write $s = \lfloor \frac{q}{3} \rfloor$. First, assume that

$$\sharp B^c \leq s(q-1) + 2q,$$

then

$$\sharp B \geq q^2 - s(q-1) - 2q \geq q^2 - \frac{q}{3}(q-1) - 2q = \frac{2q^2}{3} - \frac{5q}{3}. \tag{3.1}$$

Else suppose that

$$\sharp B^c \geq s(q-1) + 2q.$$

Since B is a Nikodym set, for each $x \in B^c$ there exists a line L_x such that

$$L_x \cap B^c = \{x\}.$$

Obviously, all of these lines are distinct from each other. Noting that there are in total $q+1$ directions in $\mathbb{F}_q^2$, we partition $\{L_x\}_{x\in B^c}$ into classes $\{G_i\}_{i=0}^q$ according to their directions. Without loss of generality we may assume that

$$\sharp G_0 \geq \sharp G_1 \geq \sharp G_2 \geq \cdots \geq \sharp G_q.$$

Thus

$$q + q + \sharp G_2 \cdot (q-1) \geq \sum_{i=0}^{q} \sharp G_i = \sharp B^c \geq s(q-1) + 2q,$$

from which yields

$$\sharp G_2 \geq s.$$

Choose s parallel lines $\{X_l\}_{l=1}^s$ from G_0, s parallel lines $\{Y_m\}_{m=1}^s$ from G_1 and s parallel lines $\{Z_n\}_{n=1}^s$ from G_2, then it follows that

$$\begin{aligned}\sharp B &\geq \sum_{l=1}^{s}(\sharp X_l - 1) + \sum_{m=1}^{s}(\sharp Y_m - 1 - s) + \sum_{n=1}^{s}(\sharp Z_n - 1 - 2s)\\ &= s(q-1) + s(q-1-s) + s(q-1-2s) = 3s(q-1-s)\\ &\geq 3\frac{q-2}{3}(q-1-\frac{q}{3}) = \frac{2q^2}{3} - \frac{7q}{3} + 2. \qquad (3.2)\end{aligned}$$

Combining (3.1) and (3.2) yields the desired result.

□

Question: How small can the Nikodym sets really be in two dimensions?

4. Acknowledgements

The author thanks Yaokun Wu for clarifying the proof. He also thanks Aart Blokhuis and Qing Xiang for kindly pointing out the recent progresses on the finite field Kakeya problem to the author.

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